Herman rings

[Carleson-Gamelin] 1987

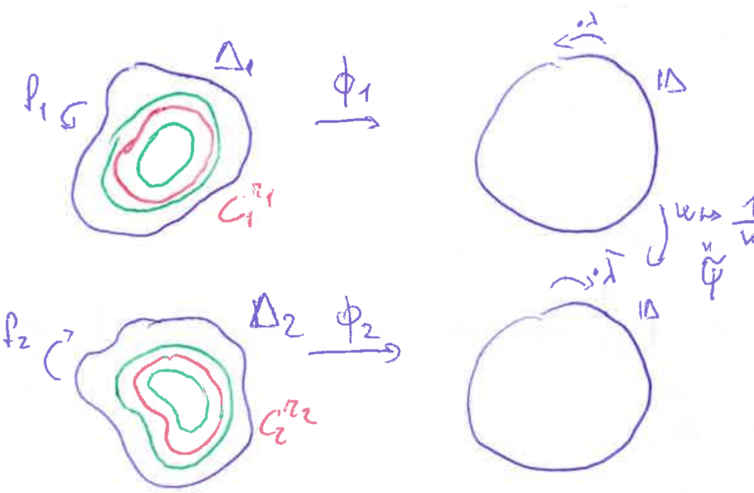
Theorem (Herman, Shishikura). Suppose a rational function $f_1 : \hat{\mathbb{C}} \circlearrowleft$ has a Siegel disk, on which f_1 is conjugate to multiplication by $\lambda = e^{2\pi i \alpha}$.
Then there exists another rational function with a Herman ring, on which the action is by multiplication by λ.

Proof. We may assume that the siegel disk Δ_1 of f_1 is centered at 0.
Consider the map $f_2(z) = \overline{f_1(\bar{z})}$, which has a siegel disk Δ_2 of multiplier $\bar{\lambda} = e^{-2\pi i \alpha}$. Consider Siegel coordinate $\phi_j : \Delta_j \to D = \Delta(0,2)$
disk, suppose of radius 2 up to dilatations.

More generally, we could work with two different maps f_1, f_2 of degree d_1, d_2 with siegel disks Δ_1, Δ_2 of multiplier $\lambda, \bar{\lambda}$.

Denote by C_j^r the invariant circle in Δ_j corresponding to $\{|w| = r\}$ in the normal form given by the Siegel coordinate ϕ_j.

Take $0 < r_1, r_2 < 2$ (pick $r_1 = r_2 = 1$) and define an orientation-reversing homeomorphism ψ from an annular neighborhood of $C_1^{r_1} =: C_1$ to an annular neighborhood of $C_2^{r_2} =: C_2$, so that, in the normal form spaces, ψ corresponds to $w \mapsto \frac{1}{w}$.
The tubular neighborhoods will be of the form $A_j = \bigcup_{\frac{1}{R} < r < R} C_j^r$, $1 < R \leq 2$ (e.g. $R = 2$)



$\psi : A_1 \to A_2,$

$\psi = \phi_2^{-1} \circ \tilde{\psi} \circ \phi_1$

In particular, $\varphi(C_1) = C_2$ and $\varphi \circ f_1 \overset{(*)}{=} f_2 \circ \varphi$ on A_1:

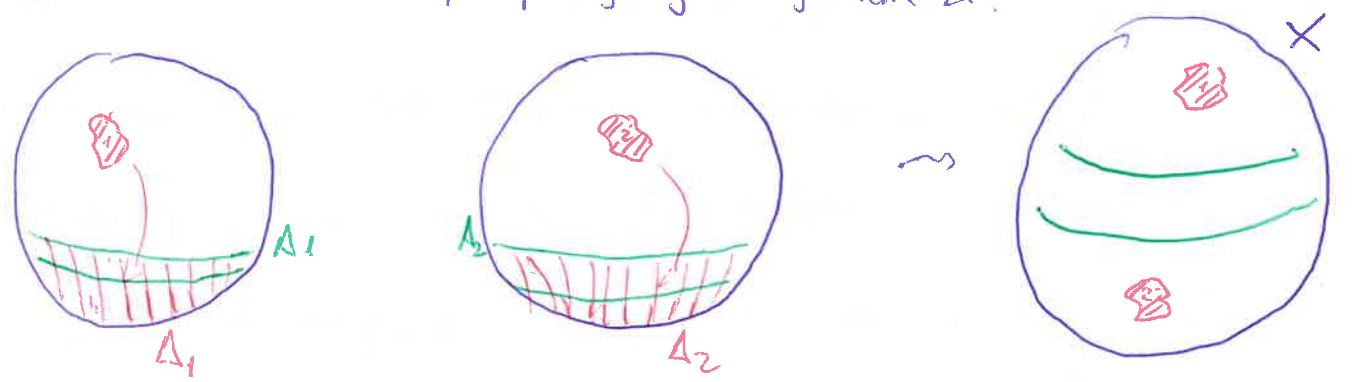
$$\varphi \circ f_1 = \phi_2^{-1} \circ \tilde{\varphi} \circ \phi_1 \circ f_1 = \phi_2^{-1} \circ \tilde{\varphi} \circ \lambda \circ \phi_1 , \qquad \tilde{\varphi} \circ \lambda(w) = \frac{1}{\lambda w} \quad))$$

$$f_2 \circ \varphi = f_2 \circ \phi_2^{-1} \circ \tilde{\varphi} \circ \phi_1 = \phi_2^{-1} \circ \bar{\lambda} \circ \tilde{\varphi} \circ \phi_1 \qquad \bar{\lambda} \circ \tilde{\varphi}(w) = \bar{\lambda} \frac{1}{w} .$$

Now, consider a new Riemann surface X obtained by gluing together $\hat{\mathbb{C}} \setminus \bar{\Delta}_1^{\frac{1}{R}}$ and $\hat{\mathbb{C}} \setminus \bar{\Delta}_2^{\frac{1}{R}}$ $\left(\Delta_j^{\frac{1}{R}} = \phi_j^{-1}(\{|w| < \tfrac{1}{R}\}) \right)$ along A_1, A_2 by the homeomorphism φ: $X = \dfrac{\hat{\mathbb{C}} \setminus \bar{\Delta}_1^{\frac{1}{R}} \sqcup \hat{\mathbb{C}} \setminus \bar{\Delta}_2^{\frac{1}{R}}}{z_2 \sim \varphi(z_1)}$

X is a Riemann surface, biholomorphic to $\hat{\mathbb{C}}$ (clearly it is compact and homeomorphic to S^2).

The mappings f_1 and f_2 induce a mapping $F : X \to X$, defined everywhere but for the copies of $f_j^{-1}(\Delta_j) \setminus \Delta_j$ in X.



(It is well defined also on the copy of A_1/A_2, because of the functional relation $(*)$)

We extend F to all X by setting: (we assume we are not in the copies of Δ_1 or Δ_2)

- On $f_1^{-1}(\Delta_1 \setminus \Delta_1^R)$ use $F \equiv f_1$
- On ~~$f_1^{-1}(A_1)$~~ $f_1^{-1}(\bar{\Delta}_1^{\frac{1}{R}})$, define F as any conformal mapping onto $\hat{\mathbb{C}} \setminus \Delta_2^{\frac{1}{R}}$ (the image of f_1 is $\bar{D}^{\frac{1}{R}}$, a closed disk biholomorphic to the complement of a open disk in $\hat{\mathbb{C}}$.

• In $W_1 = f_1^{-1}(A_1)\setminus A_1$ we define F as any diffeomorphism onto A_1 which extends the values given on the boundary.
In particular $f \not\equiv f_1$, since F is defined otherwise $\neq f_1$ on the boundary of W_1.

• Apply a similar construction on $f_2^{-1}(\Delta_2)\setminus\Delta_2$.

Notice that F is analytic on $X\setminus(W_1\cup W_2)$.

Moreover, if the orbit of a point $z\in X$ enters $W_1\cup W_2$, then it stays in $W_1\cup W_2$, and in fact, after a certain number of steps, it enters the invariant ring A (corresponding to A_1, A_2) and stays there.

Consider on $W_1\cup W_2$ the Beltrami coefficient (form) associated to F:
$\mu = \dfrac{F_{\bar z}}{F_z} = \mu_F|_{W_1\cup W_2}$. By construction, $\mu = F^*0$ on $W_1\cup W_2$, since $F: W_1\to A_1$ is μ-quasiconformal and $F: A_1\to A$ is holomorphic.

We can extend μ to the grand orbit of W_1 by ~~setting~~ pullback through F, and similarly on the grand orbit of W_2 (F acts holomorphically in such sets), and set $\mu\equiv 0$ on the complementary.
Hence we have constructed a F-invariant Beltrami coefficient μ on $\hat{\mathbb{C}}$, which coincides with μ_F on $W_1\cup W_2$.
Let $\Phi_\mu : \hat{\mathbb{C}} \circlearrowleft$ be the normalized solution of the Beltrami equation associated to μ. Then the map $\Phi_\mu\circ F\circ\Phi_\mu^{-1} =: f$ is a rational map with a Herman ring (corresponding to $\Phi_\mu(A)$).

□

Remarks: 1) We defined the pull back of a Beltrami coefficient only for holomorphic maps but we can define a pull back for general quasiconformal maps by using the interpretation of Beltrami forms (or equivalently using the formula for $\mu_{g\circ f}$)

2) In the definition of $F|_{W_1}$, we need to take the diffeomorphism quasiconformal. This is ensured by the fact that it is a diffeo preserving the orientation. (we would have a critical point if $|\mu|=1$ ...)

Remark: By construction, the number of critical points of f is the sum of the number of critical points of f_1 and f_2. Hence $\deg f = \deg f_1 + \deg f_2 - 1$.

Other results regarding Herman rings:

By means of quasiconformal surgery, one can count Herman rings:

Theorem (Shishikura): The number of Herman rings of a rational map $f: \hat{\mathbb{C}} \circlearrowleft$ of degree $d \geq 2$ is bounded by $d-2$.

This bound is sharp.

We will prove a coarser bound: $4d+2$.

The proof is similar to Sullivan non wandering domain theorem.

Let U be a Herman ring and $E \subset\subset U$ be a compact invariant subset, conformally equivalent to the annulus $\overline{\mathbb{A}}_R = \{ w \in \mathbb{C} \mid 1 \leq |w| \leq R \}$, $R>1$ modulus of $\overline{\mathbb{A}}_R \cong E$

Consider the map $\tilde{\nu}: \overline{\mathbb{A}}_R \to \partial\mathbb{D}$ given by $\tilde{\nu}(w) = \frac{w^2}{|w|^2}$, $\tilde{\nu}_t = t\tilde{\nu}$, and $\nu_t = \phi^* \tilde{\nu}_t$ as a Beltrami form, for $0<t<1$, the ellipse field corresponding to the Beltrami coefficient $\nu_t =$ is invariant under rotation, with major axis parallel to the direction of the rotation

In fact, $\arg \frac{\tilde{\nu}_t}{2} = \arg w$ is the minor axis.

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A solution of ~~the~~ the Beltrami equation associated to $\tilde{\nu}_t$ is explicitly given by $\tilde{\Phi}_t(w) = |w|^{\frac{2t}{1-t}} \cdot w$

$$\Phi_t = w^a \bar{w}^b$$

$$\frac{\frac{\partial \Phi}{\partial \bar{w}}}{\frac{\partial \Phi}{\partial w}} = \frac{b \bar{w}^{b-1} w^a}{a w^{a-1} \bar{w}^b} = \frac{b}{a} \cdot \frac{w}{\bar{w}}$$

take $\frac{b}{a} = t$. To have a homeo. we need $a = b+1 \Rightarrow a = \frac{1}{1-t}$, $b = \frac{t}{1-t}$.

Being $\frac{2t}{1-t} \geq 0$ and $|w| \geq 1$, $\tilde{\Phi}_t$ increases the length of any circle $\partial \mathbb{D}_r$, $r > 1$, and Φ_t increases the length of $\varphi^{-1}(\partial \mathbb{D}_r) =: \mathcal{C}_r$.

Consider now N such annuli associated to different Herman rings $1, \dots, N$

Define E_j, $\nu_{j,t}$ as above, and for $t \in Q_N = (0,1)^N$, define the Beltrami coefficient μ_t to be ν_{j,t_j} on E_j as above $(t_1, \dots, t_N)$

Extend μ_t to the inverse iterates of the E_j's by means of pullback through $(f^k)^*$ as Beltrami forms. Set $\mu_t \equiv 0$ on the complement of the grand orbit of $\cup E_j$.

By placing ∞ on $U_1 \setminus E_1$, μ_t has compact support in $\mathbb{C}$

Let Φ_t be the normalised solution of the Beltrami equation associated to μ_t.

The map $f_t = \Phi_t \circ f \circ \Phi_t^{-1}$ is rational of degree d, and f_t (its coefficients) vary continuously on $t \in Q_N$.

Each $\Phi_t(U_j)$ is a Herman ring for f_t, for which the modulus is a strictly increasing function of t_j.

Hence there is a open set $W \subset Q_N$ where the maps f_t are all distinct.

But $\mathrm{Rat}_d(\hat{\mathbb{C}})$ has complex dimension $2d+1$. We deduce that $N \leq 2 \cdot (2d+1)$

$\square$

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Shishikura theorem.

Another theorem proved by quasiconformal surgery is:

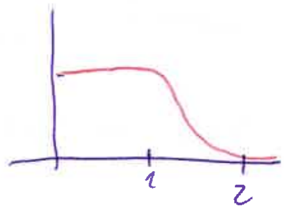
Theorem (Shishikura). $f : \hat{\mathbb{C}} \circlearrowleft$ rational map of degree $d \geq 2$ has at most $2d-2$ non-repelling cycles

Idea of the proof: Deform f to transform all non-repelling cycles in attracting cycles

Idea 1: take $h \in \mathbb{C}[z]$, $h(z) = 0$ $\forall z$ in non-repelling cycle (assume ∞ not periodic) and $h'(z) = -1$ on these points.

Deform f by $f_\varepsilon(z) = f(z + \varepsilon h(z))$, analytic, but $\deg f_\varepsilon >> \deg f$.

Idea 2: Quasiconformal deformation: $\rho : [0, +\infty) \to [0,1]$ decreasing, C^∞, $\rho(x) = \begin{cases} 1 & x \leq 1 \\ 0 & x \geq 2 \end{cases}$.



- $H_\varepsilon(z) = z + \varepsilon \rho(\varepsilon^{\frac{1}{k}} |z|) h(z)$, ← quasiconformal.

$$\underset{\substack{\uparrow \\ \text{quasiregular}}}{S_\varepsilon(z)} = f \circ H_\varepsilon(z) = \begin{cases} f(z), & |z| \geq 2\varepsilon^{-\frac{1}{k}} \\ f_\varepsilon(z), & |z| \leq \varepsilon^{-\frac{1}{k}} \leftarrow V_\varepsilon \end{cases}$$

- Apply quasiconformal surgery to S_ε to conjugate S_ε to a rational map of degree d with all original cycles being attracting

~~Rsfor~~ To do so we need to construct E so that $E \subset V_\varepsilon$, $\underset{f_\varepsilon}{\underset{\shortparallel}{S_\varepsilon}}(E) \subseteq E$ and $S_\varepsilon(V_\varepsilon^c) \subset E$. In fact this implies that:

$\hat{\mathbb{C}} \setminus S_\varepsilon^{-1}(E) \subseteq \hat{\mathbb{C}} \setminus (E \cup V_\varepsilon^c) \subseteq V_\varepsilon$, where $S_\varepsilon \equiv f_\varepsilon$ is analytical.

The construction of E is made case by case for all type of cycles.